SS 2016

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Differential Geometry

Homework 7

Mandatory Exercise 1. (10 points)
Let ∇ be an affine connection on M. If $\omega \in \Omega^1(M)$ and X a vector field on M, we define the **covariant derivative** of ω along X, $\nabla_X \omega \in \Omega^1(M)$, by

$$\nabla_X \omega(Y) = X(\omega(Y)) - \omega(\nabla_X Y)$$

for all vector fields Y.

(a) Show that this formula defines indeed a 1-form.

(b) Show that:

- $\nabla_{fX+gY}\omega = f\nabla_X \omega + g\nabla_Y \omega$
- $\nabla_X(\omega + \eta) = \nabla_X \omega + \nabla_X \eta$
- $\nabla_X(f\omega) = (Xf)\omega + f\nabla_X \omega$

for all vector fields X, Y, $f, g \in C^\infty(M)$ and $\omega, \eta \in \Omega^1(M)$.

(c) Let $x\colon W \to \mathbb{R}^n$ be local coordinates on an open set $W \subset M$, and take

$$\omega = \sum_{i=1}^{n} \omega_i \, dx^i.$$

Show that

$$\nabla_X \omega = \sum_{i=1}^{n} \big(X\omega_i - \sum_{j,k=1}^{n} \Gamma_{ij}^k X^j \omega_k \big) \, dx^i.$$

Mandatory Exercise 2. (10 points)
Let X, X', Y and Y' be vector fields on M such that $X = X'$ and $Y = Y'$ on an open set $W \subset M$. Show that

$$\nabla_X Y = \nabla_{X'} Y'.$$

Suggested Exercise 1. (0 points)
Recall that while proving the existence of Levi-Civita connection we defined it using the Koszul formula:

$$2\langle \nabla_X Y, Z\rangle = X(\langle Y, Z\rangle) + Y(\langle X, Z\rangle) - Z(\langle X, Y\rangle) - \langle [X, Z], Y\rangle - \langle [Y, Z], X\rangle + \langle [X, Y], Z\rangle.$$

Show that $\nabla_X Y$ defined this way is indeed an affine connection.

Suggested Exercise 2. (0 points)
Prove that the curvature tensor is indeed a tensor.

Suggested Exercise 3. (0 points)
Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇. Show that g is **parallel** along any curve, i.e. show for an arbitrary vector field X that

$$\nabla_X g = 0.$$

Suggested Exercise 4. (0 points)
Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇, and let X be the Killing vector field.

(a) $\langle \nabla_Y X, Z\rangle + \langle \nabla_Z X, Y\rangle = 0$ for all vector fields Y, Z.

(b) If $c\colon I \to M$ is a geodesic then $\langle \dot{c}(t), X_{c(t)}\rangle$ is constant.

Suggested Exercise 5. (0 points)
Recall that if M is an oriented differentiable manifold with volume element $\omega \in \Omega^n(M)$, the divergence of X is the function $\operatorname{div}(X)$ such that

$$\mathcal{L}_X \omega = \operatorname{div}(X)\omega.$$

Suppose that M carries a Riemannian metric and that ω is a Riemannian volume element. Show that at each point $p \in M$,

$$\operatorname{div}(X) = \sum_{i=1}^{n} \langle \nabla_{Y_i} X, Y_i\rangle,$$

where $\{Y_1, \dots, Y_n\}$ is an orthonormal basis of $T_p M$ and ∇ is the Levi-Civita connection.

Hand in: Monday 6th June
in the exercise session
in Seminar room 2, MI